

ON A NONLOCAL AGGREGATION MODEL WITH NONLINEAR DIFFUSION

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ABSTRACT. We consider a nonlocal aggregation equation with nonlinear diffusion which arises from the study of biological aggregation dynamics. As a degenerate parabolic problem, we prove the well-posedness, continuation criteria and smoothness of local solutions. For compactly supported nonnegative smooth initial data we prove that the gradient of the solution develops L_x^∞ -norm blowup in finite time.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider the following evolution equation with nonlinear diffusion:

$$\begin{cases} \partial_t u + \partial_x(u \partial_x K * u) = r \partial_x(u^2 \partial_x u), & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where K is an even function of x and has a Lipschitz point at the origin, e.g. $K(x) = e^{-|x|}$. Here $*$ denotes the usual spatial convolution on \mathbb{R} . The parameter $r > 0$ measures the strength of the nonlinear diffusion term. The function $u = u(t, x)$ represents population density in biology or particle density in material science. (see, for example, [56] [26] [30] [31] [46] [57] [40] [58] [27] [47] [51]). This equation was used in the study of biological aggregation such as insect swarms, fish schools and bacterial colonies. It is first derived by Bertozzi, Lewis and Topaz [56] as a modification (more precisely, the addition of the density-dependent term on the RHS of (1.1)) of an earlier classical model of Kawasaki [35]. According to [56], this modification gives rise to biologically meaningful clumping solutions (i.e. densities with compact support and sharp edges). For other similar one-dimensional models and their biological applications we refer the readers to [46] [26] [60] [55] [29] [32] [33] [34] [35] [49] [50] and the references therein for more extensive background and reviews.

To understand the biological meaning of each term, one can rewrite (1.1) as the classical continuity equation

$$\partial_t u + \partial_x(uv) = 0, \quad (1.2)$$

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where the velocity v is related to the density u by

$$\begin{aligned} v &= \partial_x(K * u) - ru\partial_x u \\ &=: v_a + v_d. \end{aligned}$$

Here v_a is called the attractive velocity since as explained in [56] individuals aggregate by climbing gradients of the sensing function $s = K * u$. Due to the spatial convolution v_a is a nonlocal transformation of the density u . The second term v_d is called the dispersal(repulsive) velocity and is a spatially local function of both the population and the population gradient. Biologically v_d represents the anti-crowding mechanism which operates in the opposite direction of population gradient and decreases as the population density drops. These constitutive relations of v_a , v_d , u are natural in view of the basic biological assumption that aggregation occurs on a longer spatial scale than dispersal.

In the mathematics literature, the aggregation equations which have similar forms to (1.1) have been studied extensively [11] [12] [13] [14] [15] [38] [57]. The case of (1.1) with $r = 0$ and general choices of the kernel K was considered by Bodnar and Velazquez [13]. There by an ODE argument the authors proved the local well-posedness of (1.1) *without the density-dependent term* for C^1 initial data. For a generic class of choices of the kernel K and initial data, they proved by comparing with a Burgers-like dynamics, the finite time blowup of the L_x^∞ -norm of the solution. Burger and Di Francesco [14] studied a class of one-dimensional aggregation equations of the form

$$\partial_t u = \partial_x (u \partial_x (a(u) - K * u + V)), \quad \text{in } (0, \infty) \times \mathbb{R},$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$ is a given external potential and the nonlinear diffusion term $a(\rho)$ is assumed to be either 0 or a strictly increasing function of ρ . In the case of no diffusion ($a \equiv 0$) they proved the existence of stationary solutions and investigated the weak convergence of solutions toward the steady state. In the case of sufficiently small diffusion ($a(\rho) = \epsilon \rho^2$) they proved the existence of stationary solutions with small support. Burger, Capasso and Morale [15] studied the well-posedness of an equation similar to (1.1) but with a different diffusion term:

$$\partial_t u + \nabla \cdot (u \nabla K * u) = \text{div}(u \nabla u), \quad \text{in } (0, T) \times \mathbb{R}^d.$$

For initial data $u_0 \in L_x^1(\mathbb{R}^d) \cap L_x^\infty(\mathbb{R}^d)$ with $u_0^2 \in H_x^1(\mathbb{R}^d)$, they proved the existence of a weak solution by using the standard Schauder's method. Moreover the uniqueness of entropy solutions was also proved there. In connection with the problem we study here, Laurent [38] has studied in detail the case of (1.1) *without density-dependent diffusion* (i.e. $r = 0$) and proved local and global existence results for a class of kernels K with $H_x^s(s \geq 1)$ initial data. More recently Bertozzi and Laurent [11] have obtained finite-time blowup of solutions for the case

of (1.1) without diffusion (i.e. $r = 0$) in $\mathbb{R}^d (d \geq 2)$ assuming compactly supported radial initial data with highly localized support. Li and Rodrigo [41] [42] [43] studied the case of (1.1) with fractional dissipation and proved finite-time blowup or global wellposedness in various situations. We refer to [44] [45] for the cases with singular kernels and further detailed studies concerning sharp asymptotics and regularity of solutions. We also mention that Bertozzi and Brandman [7] recently constructed $L_x^1 \cap L_x^\infty$ weak solutions to (1.1) in $\mathbb{R}^d (d \geq 2)$ and with no dissipation ($r = 0$) by following Yudovich's work on incompressible Euler equations [61]. We refer the interested readers to [55] [29] [32] [33] [34] [35] [49] [50] and the references therein for some further rigorous studies.

From the analysis point of view, equation (1.1) is also connected with a general class of degenerate parabolic equations known as porous medium equations, which takes the form

$$\partial_t u = \partial_x (u^m \partial_x u), \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (1.3)$$

where m is a real number. These equations describe the ideal gas flow through a homogeneous porous medium and other physical phenomena in gas dynamics and plasma physics [48] [62] [5]. Oleinik, Kalashnikov and Chzhou [52] proved the existence and uniqueness of a global weak solution of (1.3). Hölder continuity of the weak solution is established in [4] [37] [28] [20]. As for higher regularities, the Lipschitz continuity of the pressure variable $v = \frac{m+1}{m}u$ is shown in [4] [25] [10] [2] [24]. Aronson and Vazquez [6] showed that v becomes C^∞ outside of the free boundary after some waiting time. For derivative estimates of the solution u , a local solution of (1.3) in $W_x^{1,\infty}(\mathbb{R})$ is constructed by Otani and Sugiyama [53]. In the case of (1.3) with m being an even natural number, Otani and Sugiyama [54] proved the existence of smooth solutions. We refer the interested readers to [1] [3] [9] [8] [16] [17] [21] [23] [24] [18] [19] [22] [36] [59] and references therein for more detailed studies and expositions.

Our starting point of the analysis of equation (1.1) is to treat it as a degenerate parabolic problem with a nonlocal flux term. We focus on constructing and analyzing classical solutions of (1.1). The bulk of this paper is devoted to proving the existence and uniqueness of classical solutions to (1.1), which is done in section 2 and part of section 3. In the final part of section 3, we prove the continuation and blowup criteria of solutions. In the last section we prove that any smooth initial data with compact support will lead to blowup of the gradient in finite time. The analysis developed in this work can be extended to treat the d -dimensional ($d \geq 2$) case of (1.1) which we will address in a future publication.

We now state more precisely our main results. The first theorem establishes the existence of smooth local solutions for initial data which is not necessarily nonnegative.

Theorem 1.1 (Existence and uniqueness of smooth local solution). *Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ (u_0 is not necessarily nonnegative). Then there exists a positive $T_0 = T_0(\|u_0\|_2 + \|\partial_x u_0\|_{\infty})$ such that (1.1) has a unique solution $u \in C^{\infty}([0, T_0] \times \mathbb{R})$. In particular $u \in C([0, T_0] \cap H_x^k)$ for any $k \geq 0$. If $u_0 \geq 0$, then $u(t) \geq 0$ for all $0 \leq t \leq T_0$.*

Remark 1.2. The assumptions on the initial data u_0 in Theorem 1.1 can be weakened significantly (see for example Theorem 3.2). However in order to simplify the presentation, we do not state our theorems here in its most general form.

Our second theorem gives the blowup or continuation criteria of solutions. Roughly speaking, it says that all the L_x^p -norm of the solution cannot blow up and we can continue the solution as long as we have a control of the gradient of the solution.

Theorem 1.3 (Blowup alternative and growth estimate of L_x^p -norm). *Assume $u \in C^{\infty}([0, T) \times \mathbb{R})$ is a maximal-lifespan solution obtained in Theorem 1.1. Then either $T = +\infty$ or $T < +\infty$ and*

$$\lim_{t \rightarrow T} \|\partial_x u(t)\|_{\infty} = +\infty.$$

For any $2 \leq p < \infty$, there exists a generic constant C such that

$$\|u(t)\|_p \leq \|u_0\|_p e^{Cpt}, \quad \forall t \in [0, T).$$

If in addition $u_0 \geq 0$, then $u(t) \geq 0$ for all $t \in [0, T)$, and we also have the p -independent estimate for all $2 \leq p \leq +\infty$:

$$\|u(t)\|_p \leq \|u_0\|_p \exp(Ct\|u_0\|_2 e^{Ct}), \quad \forall t \in [0, T).$$

In particular if $p = +\infty$, then

$$\|u(t)\|_{\infty} \leq \|u_0\|_{\infty} \exp(Ct\|u_0\|_2 e^{Ct}), \quad \forall t \in [0, T).$$

The next theorem states that if we assume the initial data has a little bit more integrability, then the local solution will inherit this property. Note in particular that the L_x^1 -norm of the solution is preserved for all time if the initial data is nonnegative and in $L_x^1(\mathbb{R})$.

Theorem 1.4 (Local solution in $L_x^p(\mathbb{R})$). *Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ and $u_0 \in L_x^p(\mathbb{R})$ for some $1 \leq p < 2$. Then the local solution obtained in Theorem 1.1 also satisfies $u \in C([0, T], L_x^p)$. If in addition $u_0 \geq 0$, then $u(t) \geq 0$ for any $t \in [0, T]$. If also $p = 1$, then $\|u(t)\|_1 = \|u_0\|_1$, i.e. L_x^1 -norm of the solution is preserved.*

The last theorem states that any solution with smooth nonnegative initial data will blow up in finite time.

Theorem 1.5 (Finite time blowup). *Let $u_0 \in C_c^\infty(\mathbb{R})$ and $u_0 \geq 0$. Assume u_0 is not identically 0. Then there exists time $T < \infty$ such that the corresponding solution with u_0 as initial data blows up at time T in the sense that*

$$\lim_{t \rightarrow T} \|\partial_x u(t)\|_\infty = +\infty. \quad (1.4)$$

However all the L_x^p -norm of u remain finite at the time of blowup, i.e.:

$$\sup_{0 \leq t < T} \|u(t)\|_p \leq C(T) < \infty, \forall 1 \leq p \leq \infty. \quad (1.5)$$

Remark 1.6. In the case of (1.1) *without diffusion* (i.e. $r = 0$ in (1.1)), if we take the kernel $K(x) = e^{-|x|}$, then the result of Bodnar and Velazquez [13] says that any solution with initial data u_0 satisfying a slope condition will blow up in finite time in the sense that $\|u(t)\|_\infty$ blows up. This is highly in contrast with our result here when the diffusion term does not vanish. In this case the solution will blow up at the level of the gradient, i.e. we have $\|\nabla u(t)\|_\infty$ tends to infinity while all the other L_x^p -norm remain finite as $t \rightarrow T$, where T is the blowup time.

Notations. Throughout the paper we denote $L_x^p = L_x^p(\mathbb{R})$ for $1 \leq p \leq \infty$ as the usual Lebesgue space on \mathbb{R} . We also write $\|\cdot\|_p = \|\cdot\|_{L_x^p}$. For $s > 0$, s being an integer and $1 \leq p \leq \infty$, $W_x^{s,p} = W_x^{s,p}(\mathbb{R})$ denotes the usual Sobolev space

$$W_x^{s,p} = \{f \in S'(\mathbb{R}) : \|f\|_{W^{s,p}} = \sum_{0 \leq j \leq s} \|\partial_x^j f\|_p < \infty\}.$$

When $p = 2$, we denote $H_x^m = H_x^m(\mathbb{R}) = W_x^{2,p}(\mathbb{R})$ and $\|\cdot\|_{H^m}$ as its norm. Occasionally we shall use the Sobolev space of fractional power $H_x^s(\mathbb{R})$ whose norm can be defined via Fourier transform:

$$\|f\|_{H^s} = \|(1 + |\xi|)^s \hat{f}(\xi)\|_{L_\xi^2}.$$

For any two quantities X and Y , we use $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some constant $C > 0$. A constant C with subscripts implies the dependence on these parameters. We write $A = A(B_1, \dots, B_k)$ when we want to stress that a quantity A depends on the quantities B_1, \dots, B_k .

From now on we assume $r = 1$ in (1.1) without loss of generality. Same results hold for any $r > 0$.

2. THE REGULARIZED EQUATION AND ITS WELLPOSEDNESS

Since (1.1) is a degenerate parabolic equation, in order to construct a local solution, we have to regularize the equation. To this end, we

consider the following regularized version of (1.1)

$$\begin{cases} \partial_t u = (u^2 + \epsilon) \partial_{xx} u + 2u(\partial_x u)^2 - \partial_x(u K * \partial_x u), \\ u(0, x) = u_0(x). \end{cases} \quad (2.1)$$

Here $\epsilon > 0$ is a parameter. We are going to prove the following

Proposition 2.1 (Local solution of the regularized equation). *Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ (u_0 is not necessarily nonnegative). Then there exists a positive $T_0 = T_0(\|\partial_x u_0\|_{\infty} + \|u_0\|_2)$ (T_0 is independent of ϵ) such that (2.1) has a unique solution $u^{\epsilon} \in C^{\infty}([0, T_0] \times \mathbb{R})$ for any $\epsilon > 0$. In particular $u^{\epsilon} \in C([0, T_0]; H_x^k)$ for any $k \geq 0$.*

Corollary 2.2. *Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ and $u_0 \in L_x^p(\mathbb{R})$ for some $1 \leq p < 2$. Then the local solution obtained in Proposition 2.1 also satisfies $u^{\epsilon} \in C([0, T_0], L_x^p)$. And T_0 can be chosen to be $T_0 = T_0(\|u_0\|_p + \|\partial_x u_0\|_{\infty})$.*

In the case $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ with $u_0 \geq 0$. The regularized equation admits a global solution.

Proposition 2.3. *Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R})$ with $u_0 \geq 0$. Then for any $T > 0$ (2.1) has a unique solution $u^{\epsilon} \in C^{\infty}([0, T] \times \mathbb{R})$. In particular $u^{\epsilon} \in C([0, T]; H_x^k)$ for any $k \geq 0$. Also $u^{\epsilon}(t) \geq 0$ for any $t \geq 0$. If in addition $u_0 \in L_x^p(\mathbb{R})$ for some $1 \leq p < 2$, then $u^{\epsilon} \in C([0, T]; L_x^p)$.*

2.1. Proof of Proposition 2.1. Our proof of Proposition 2.1 is reminiscent of the L_x^{∞} energy method used by Otani and Sugiyama [54] where they dealt with the porous medium equation (1.3). Denote the set

$$B_T^k = \{v \in C([0, T]; H_x^{2k+1}(\mathbb{R})) : \partial_{xx} v, \partial_t v \in L^2([0, T]; H_x^{2k}(\mathbb{R}))\}.$$

As a very first step, we shall show the local existence of the solution to (2.1) in B_T^k . At this point, we need the following lemma from [54].

Lemma 2.4. *Consider the initial value problem*

$$\begin{cases} \partial_t u = (u^2 + \epsilon) \partial_{xx} u + h(t, x), \\ u(0) = u_0, \end{cases} \quad (2.2)$$

where $h \in L^2(0, T; H_x^{2k})$, $u_0 \in H_x^{2k+1}(\mathbb{R})$, there exists a unique function $u = u(t, x) \in B_T^k$ which satisfies, for some constant $C_1 = C_1(\|u_0\|_{H_x^{2k+1}}, \|h\|_{L_t^2 H_x^{2k}}, k, \epsilon)$, that

$$\|u\|_{L_t^{\infty} H_x^{2k+1}} + \epsilon^{\frac{1}{2}} \|u\|_{L_t^2 H_x^{2k+2}} \leq C_1.$$

For any $h_1, h_2 \in K_R^T = \{v : \|v\|_{L^2(0, T; H_x^{2k})} \leq R\}$, there exists a constant $C_2 = C_2(R, k, \epsilon)$ such that

$$\|u_1 - u_2\|_{L_t^{\infty} H_x^{2k+1}} + \epsilon^{\frac{1}{2}} \|u_1 - u_2\|_{L_t^2 H_x^{2k+2}} \leq C_2 e^{C_2 T} \|h_1 - h_2\|_{L_t^2 H_x^{2k}}.$$

Proof. See [54]. □

Based on this lemma, we establish the local existence of solutions of the regularized equation. Note however that the time of existence of the solution depends on the regularization parameter ϵ . We have the following

Lemma 2.5 (Local existence of the regularized equation). *Let $u_0 \in H_x^{2k+1}(\mathbb{R})$, $k \geq 2$. There exists $T_0 = T_0(\|u_0\|_{H_x^{2k+1}}, k, \epsilon) > 0$, such that (2.1) has a unique solution $u \in B_{T_0}^k$.*

Proof. Introduce the auxiliary equation

$$\begin{cases} \partial_t u = (u^2 + \epsilon) \partial_{xx} u + h(t, x) \\ u(0) = u_0 \end{cases} \quad (2.3)$$

By Lemma 2.4, for any $u_0 \in H_x^{2k+1}(\mathbb{R})$, $h \in L^2(0, T; H_x^{2k})$, (2.3) has a unique solution $u_h \in B_T^k$. Next define the map

$$\phi(h) = 2u_h(\partial_x u_h)^2 - \partial_x(u_h K * \partial_x u_h).$$

It suffices to show that for some suitable R and T_0 , ϕ is a contraction from the set $K_R^{T_0} = \{v \in L^2(0, T_0; H_x^{2k}); \|v\|_{L^2(0, T_0; H_x^{2k})} \leq R\}$ into itself. First we show that ϕ maps $K_R^{T_0}$ into $K_R^{T_0}$. Take any $h \in K_R^T$, by the fact that $H_x^s(\mathbb{R})$ is an algebra when $s > \frac{1}{2}$, we have

$$\begin{aligned} \|\phi(h)\|_{H_x^{2k}} &\leq \|u_h\|_{H_x^{2k}} \|u_h\|_{H_x^{2k+1}}^2 + \|\partial_x K * \partial_x u_h u_h\|_{H_x^{2k}} + \|K * \partial_x u_h \partial_x u_h\|_{H_x^{2k}} \\ &\leq C(\|u_h\|_{H_x^{2k+1}}^3 + \|u_h\|_{H_x^{2k+1}}^2), \end{aligned}$$

where C is a generic constant. Then Lemma 2.4 implies that

$$\begin{aligned} \|\phi(h)\|_{L^2(0, T; H_x^{2k})} &\leq CT^{\frac{1}{2}}(\|u_h\|_{L_t^\infty(0, T; H_x^{2k+1})}^3 + \|u_h\|_{L_t^\infty(0, T; H_x^{2k+1})}^2) \\ &\leq T^{\frac{1}{2}} C_3(\|u_0\|_{H_x^{2k+1}}, R, k, \epsilon). \end{aligned}$$

This shows that if we take $T \leq C_3^2/R^2$, then ϕ maps K_R^T into K_R^T . Similarly we can show that ϕ is Lipschitz, i.e. for any $h_1, h_2 \in K_R^T$, we have

$$\|\phi(h_1) - \phi(h_2)\|_{L^2(0, T; H_x^{2k})} \leq C_4 T^{\frac{1}{2}} e^{C_4 T} \|h_1 - h_2\|_{L^2(0, T; H_x^{2k})},$$

where $C_4 = C_4(\|u_0\|_{H_x^{2k+1}}, R, k, \epsilon)$. Therefore if T is sufficiently small, then ϕ is a contraction on K_R^T . The lemma is proved. \square

Lemma 2.5 is not satisfactory since the time of existence of the solution depends on the regularization parameter ϵ . The following lemma removes this dependence, and at the same time, weakens the dependence on the initial norm.

Lemma 2.6. *Let $u_0 \in H_x^{2k+1}(\mathbb{R})$, $k \geq 2$. Then there exists $T_0 = T_0(\|\partial_x u_0\|_\infty + \|u_0\|_2)$ such that for any $\epsilon > 0$, (2.1) has a unique solution $u^\epsilon \in B_{T_0}^k$. If $u_0 \geq 0$, then $u^\epsilon(t) \geq 0$ for any $0 \leq t \leq T_0$.*

Proof. By Lemma 2.5, we can continue the solution as long as we can control the H_x^{2k+1} -norm of u^ϵ . In the following we give the a priori control of u^ϵ . As we will see, the H_x^{2k+1} -norm of u^ϵ will stay bounded on a time interval $[0, T_0]$ for a certain small T_0 depending only on the norms $\|\partial_x u_0\|_\infty, \|u_0\|_2$. We are then left with the task of estimating the various Sobolev norms which we will do in several steps. For simplicity of notations we shall write u^ϵ as u throughout this proof.

Step 1: L_x^p -norm estimate for $2 \leq p \leq +\infty$. Multiply both sides of (2.1) by $|u|^{p-2}u$ and integrate over x , we have

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p &= -\epsilon(p-1) \int |u|^{p-2} u_x^2 dx - (p-1) \int u_x^2 |u|^p dx \\
&\quad + (p-1) \int (\partial_x K * u) u |u|^{p-2} \partial_x u dx \\
&\leq -(p-1) \int |u|^{p-2} \left(u u_x - \frac{\partial_x K * u}{2} \right)^2 dx \\
&\quad + \frac{p-1}{4} \int |u|^{p-2} (\partial_x K * u)^2 dx \\
&\leq \frac{p-1}{4} \|\partial_x K * u\|_p^2 \|u\|_p^{p-2} \\
&\leq \frac{p-1}{4} \|\partial_x K\|_1^2 \|u\|_p^p.
\end{aligned} \tag{2.4}$$

Gronwall's inequality implies that

$$\|u(t)\|_p \leq \|u_0\|_p \cdot \exp \left(\frac{p-1}{4} \|\partial_x K\|_1^2 t \right), \quad \forall 2 \leq p < +\infty. \tag{2.5}$$

In particular we have

$$\|u(t)\|_2 \leq \|u_0\|_2 \exp \left(\frac{1}{4} \|\partial_x K\|_1^2 t \right). \tag{2.6}$$

To control L_x^∞ -norm, we are going to choose $T_0 < \frac{1}{\|\partial_x K\|_1}$. This implies that $\|u(t)\|_2$ has a uniform bound on any such $[0, T_0]$. Now use again the RHS of (2.4) to get

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p &\leq (p-1) \int (\partial_x K * u) u |u|^{p-2} \partial_x u dx \\
&= \frac{p-1}{p} \int (\partial_x K * u) \partial_x (|u|^p) dx \\
&= -\frac{p-1}{p} \int (\partial_{xx} K * u) |u|^p dx.
\end{aligned}$$

Now we use the fact that $K(x) = e^{-|x|}$ and therefore $\partial_{xx}K(x) = -2\delta(x) + e^{-|x|}$, and this gives

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p &\leq -\frac{2(p-1)}{p} \int u|u|^p dx + \frac{p-1}{p} \int (e^{-|x|} * u)|u|^p dx \\ &\leq \frac{2(p-1)}{p} \|u(t)\|_{p+1}^{p+1} + \frac{p-1}{p} \|u(t)\|_p^p \|e^{-|x|}\|_2 \|u\|_2 \\ &\lesssim \frac{2(p-1)}{p} \|u(t)\|_{p+1}^{p+1} + \frac{p-1}{p} \|u(t)\|_p^p \\ &\lesssim \frac{2(p-1)}{p} \|u\|_p^p \|u\|_\infty + \frac{p-1}{p} \|u\|_p^p. \end{aligned}$$

This implies

$$\frac{d}{dt} \|u(t)\|_p \lesssim \frac{2(p-1)}{p} \|u\|_p \|u\|_\infty + \frac{p-1}{p} \|u(t)\|_p.$$

Integrating over $[0, t]$, we obtain

$$\|u(t)\|_p \leq \|u_0\|_p + C \int_0^t (1 + \|u(s)\|_\infty) \|u(s)\|_p ds.$$

Letting $p \rightarrow \infty$, we obtain

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + C \int_0^t (1 + \|u(s)\|_\infty) \|u(s)\|_\infty ds.$$

Now it's easy to see that if we choose T_0 sufficiently small depending on $\|u_0\|_2, \|u_0\|_\infty$, then we have

$$\sup_{0 \leq t \leq T_0} \|u(t)\|_r \leq C \|u_0\|_r, \quad \forall 2 \leq r \leq \infty. \quad (2.7)$$

Step 2: Control of $\|\partial_x u(t)\|_\infty$. At this point, one can appeal directly to the maximum principle (see for example Theorem 11.16 of [39]). We give a rather direct estimate here without using the maximum principle. Differentiating both sides of (2.1) w.r.t x and denoting $v = u_x$, we have

$$\partial_t v = \epsilon v_{xx} + u^2 v_{xx} + 6uvv_x + 2v^3 + G, \quad (2.8)$$

where

$$G = \partial_{xx}(uK * v).$$

Multiplying both sides of (2.8) by $|v|^{p-2}v$ and integrating over x , we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p &= \int u^2 v_{xx} |v|^{p-2} v dx + \epsilon \int v_{xx} |v|^{p-2} v dx + 6 \int uv_x |v|^{p-2} v dx \\ &\quad + 2 \int |v|^{p+2} dx + \int G(x) |v|^{p-2} v dx. \end{aligned} \quad (2.9)$$

Integrate by parts and we have

$$\begin{aligned}\int u^2 v_{xx} |v|^{p-2} v dx &= -2 \int u v_x |v|^p dx - (p-1) \int u^2 v_x^2 |v|^{p-2} dx, \\ \int u v_x |v|^p dx &= -\frac{1}{1+p} \int |v|^{p+2} dx \leq 0, \\ \epsilon \int v_{xx} |v|^{p-2} v dx &= -\epsilon(p-1) \int |v|^{p-2} v_x^2 dx \leq 0.\end{aligned}$$

Plugging the above estimates into (2.9), we get

$$\frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p \leq 2 \int |v|^{p+2} dx + \int G(x) |v|^{p-2} v dx.$$

Now we compute

$$\begin{aligned}\int G |v|^{p-2} v dx &= \int \partial_{xx} (K * v u) |v|^{p-2} v dx \\ &= \int \partial_{xx} K * v u |v|^{p-2} v dx + 2 \int \partial_x K * v |v|^p dx + \int K * v v_x |v|^{p-2} v dx \\ &\lesssim \int |u| |v|^p dx + \int (e^{-|x|} * v) u |v|^{p-2} v dx + \|v\|_p^p \|v\|_\infty \\ &\lesssim \|v\|_p^p (\|u\|_\infty + \|v\|_\infty).\end{aligned}$$

Finally we obtain

$$\begin{aligned}\frac{1}{p} \frac{d}{dt} \|v(t)\|_p^p &\lesssim \|v\|_{p+2}^{p+2} + \|v\|_p^p (\|u\|_\infty + \|v\|_\infty) \\ &\lesssim \|v\|_p^p (\|v\|_\infty^2 + \|v\|_\infty + \|u\|_\infty).\end{aligned}$$

Integrating over t , letting $p \rightarrow \infty$, and recalling $v = u_x$, we have

$$\sup_{0 \leq t \leq T_0} \|\partial_x u(t)\|_\infty \leq C \|\partial_x u_0\|_\infty, \quad (2.10)$$

for sufficiently small $T_0 = T_0(\|u_0\|_\infty, \|u_0\|_2, \|\partial_x u_0\|_\infty)$.

Step 3: Control of $\|\partial_{xx} u(t)\|_p$ for $2 \leq p \leq +\infty$. First take $2 \leq p < +\infty$ and compute

$$\begin{aligned}&\frac{1}{p} \frac{d}{dt} \int |\partial_{xx} u|^p dx \\ &= \frac{1}{3} \int |\partial_{xx} u|^{p-2} \partial_{xx} u \cdot (u^3)_{xxxx} dx + \epsilon \int |\partial_{xx} u|^{p-2} \partial_{xx} u u_{xxxx} dx \\ &\quad - \int |\partial_{xx} u|^{p-2} \partial_{xx} u \partial_{xxx} ((\partial_x K * u) u) dx.\end{aligned}$$

We further estimate by using integration by parts,

$$\begin{aligned}
& \int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot (u^3)_{xxxx} dx \\
&= 3 \int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot u^2 \partial_{xxxx}u dx + 24 \int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot u \partial_x u \partial_{xxx}u dx \\
&\quad + 18 \int |\partial_{xx}u|^p u \partial_{xx}u dx + 36 \int |\partial_{xx}u|^p (\partial_x u)^2 dx \\
&= -3(p-1) \int (\partial_{xxx}u)^2 |\partial_{xx}u|^{p-2} u^2 dx + 18(1-p) \int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot u \partial_x u \partial_{xxx}u dx \\
&\quad + 18 \int |\partial_{xx}u|^p (\partial_x u)^2 dx. \tag{2.11}
\end{aligned}$$

Using Cauchy Schwartz inequality:

$$ab \leq \frac{1}{6}a^2 + \frac{3}{2}b^2,$$

we bound the second term in (2.11) as

$$\begin{aligned}
& |18(1-p) \int |u_{xx}|^{p-2} \partial_{xx}u u \partial_x u \partial_{xxx}u dx| \\
&\leq 27(p-1) \int |u_{xx}|^p |\partial_x u|^2 dx + 3(p-1) \int |u_{xx}|^{p-2} u^2 |\partial_{xxx}u|^2 dx.
\end{aligned}$$

Hence,

$$\int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot (u^3)_{xxxx} dx \leq (27p-9) \int |\partial_{xx}u|^p (\partial_x u)^2 dx.$$

Also it is obvious that

$$\begin{aligned}
& \epsilon \int |\partial_{xx}u|^{p-2} \partial_{xx}u u u_{xxxx} dx \\
&= -\epsilon p \int (\partial_{xxx}u)^2 |\partial_{xx}u|^{p-2} dx \leq 0.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& \int |\partial_{xx}u|^{p-2} \partial_{xx}u \cdot \partial_{xxx}((\partial_x K * u)u) dx \\
&\lesssim \int |\partial_{xx}u|^{p-2} \partial_{xx}u \partial_{xxx}u (\partial_x K * u) dx + \int |\partial_{xx}u|^p \partial_x K * \partial_x u dx \\
&\quad + \int |\partial_{xx}u|^{p-2} \partial_{xx}u \partial_x u \partial_x K * \partial_{xx}u dx + \int |\partial_{xx}u|^{p-2} \partial_{xx}u u \partial_{xx}K * \partial_{xx}u dx \\
&\lesssim \int |\partial_{xx}u|^p \partial_x K * \partial_x u dx + \int |\partial_{xx}u|^{p-2} \partial_{xx}u \partial_x u \partial_x K * \partial_{xx}u dx \\
&\quad + \int |\partial_{xx}u|^p u dx + \int |\partial_{xx}u|^{p-2} \partial_{xx}u u (e^{-|x|} * \partial_{xx}u) dx \\
&\lesssim \|\partial_{xx}u\|_p^p \cdot (\|\partial_x u\|_\infty + \|u\|_\infty).
\end{aligned}$$

Collecting all the estimates and we have

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\partial_{xx} u(t)\|_p^p &\lesssim \|\partial_{xx} u\|_p^p \|\partial_x u\|_\infty^2 + \|\partial_{xx} u\|_p^p (\|\partial_x u\|_\infty^2 + \|\partial_x u\|_\infty + \|u\|_\infty) \\ &\lesssim \|\partial_{xx} u\|_p^p (\|\partial_x u\|_\infty^2 + \|\partial_x u\|_\infty + \|u\|_\infty). \end{aligned}$$

Integrating over t we have

$$\sup_{0 \leq t \leq T_0} \|\partial_{xx} u(t)\|_p \leq e^{CpT_0} \|\partial_{xx} u_0\|_p. \quad (2.12)$$

To estimate $\|u_{xx}\|_\infty$ we need to estimate $\|u_{xxx}\|_2$. We have

$$\begin{aligned} \frac{d}{dt} \|u_{xxx}(t)\|_2^2 &\lesssim - \int u_{xxxx}(u^3)_{xxxx} dx + \epsilon \int u_{xxxxx} u_{xxx} dx - \int \partial_{xxxx} (\partial_x K * uu) u_{xxx} dx \\ &\lesssim - \int u_{xxxx}^2 u^2 dx + \int u_{xxxx} (u u_x u_{xxx} + u_{xx} u_x^2 + u_{xx}^2 u) dx \\ &\quad - \int \partial_{xxxx} (\partial_x K * uu) u_{xxx} dx \\ &\lesssim \int (u_{xxx}^2 u_x^2 + u_{xx}^4 + u_{xx}^2 u_x^2) dx - \int \partial_{xxxx} (\partial_x K * uu) u_{xxx} dx. \end{aligned}$$

Since

$$\begin{aligned} &\int \partial_{xxxx} ((\partial_x K * u)u) u_{xxx} dx \\ &\lesssim \int (\partial_x K * \partial_x u) u_{xxx}^2 dx + \int (\partial_x K * \partial_{xx} u) \partial_{xx} u \partial_{xxx} u dx \\ &\quad + \int (\partial_x K * \partial_{xxx} u) u u_{xxx} dx + \int u (\partial_{xx} K * \partial_{xxx} u) u_{xxx} dx \\ &\lesssim \|u_{xxx}\|_2^2 \|\partial_x u\|_\infty + \|u_{xxx}\|_2 \|\partial_{xx} u\|_2^2 + \|\partial_{xxx} u\|_2^2 \|u\|_\infty. \end{aligned}$$

Finally we get

$$\begin{aligned} \frac{d}{dt} \|u_{xxx}(t)\|_2^2 &\lesssim \|u_{xxx}\|_2^2 (\|u_x\|_2^2 + \|u_x\|_\infty + \|u\|_\infty) \\ &\quad + \|u_{xxx}\|_2 \|\partial_{xx} u\|_2^2 + \|u_{xx}\|_4^4 + \|u_{xx}\|_2^2 \|u_x\|_\infty^2. \end{aligned}$$

Gronwall then implies that

$$\sup_{0 \leq t \leq T} \|\partial_{xxx} u(t)\|_2 \leq C \|\partial_{xxx} u_0\|_2. \quad (2.13)$$

Since $\|u_{xx}\|_\infty \lesssim \|u\|_2^{\frac{1}{6}} \|\partial_{xxx} u\|_2^{\frac{5}{6}}$, we conclude that

$$\sup_{0 \leq t \leq T_0} \|u_{xx}(t)\|_\infty \leq C.$$

Step 4: Control of $\|\partial_x^{2k+1}u\|_2$. We compute

$$\begin{aligned} \frac{d}{dt}\|\partial_x^{2k+1}u\|_2^2 &= \frac{1}{3} \int \partial_x^{2k+3}(u^3)\partial_x^{2k+1}u dx + \epsilon \int \partial_x^{2k+3}u\partial_x^{2k+1}u dx \\ &\quad - \int \partial_x^{2k+2}(K * \partial_x uu)\partial_x^{2k+1}u dx \\ &=: I + II + III \end{aligned}$$

For I , we integrate by parts and obtain

$$\begin{aligned} I &= -\frac{1}{3} \int \partial_x^{2k+2}(u^3)\partial_x^{2k+2}u dx \\ &= - \int u^2(\partial_x^{2k+2}u)^2 dx - C \int uu_x\partial_x^{2k+1}u\partial_x^{2k+2}u dx \\ &\quad + \sum_{\substack{l+m+n=2k+2 \\ l \leq m \leq n \leq 2k}} C_{l,m,n} \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+2}u dx \\ &= - \int u^2(\partial_x^{2k+2}u)^2 dx + \frac{C}{2} \int (u_x^2 + uu_{xx})(\partial_x^{2k+1}u)^2 dx \end{aligned} \quad (2.14)$$

$$+ \sum_{\substack{l+m+n=2k+3 \\ l \leq m \leq n \leq 2k+1}} C_{l,m,n} \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1}u dx \quad (2.15)$$

Discarding the negative term and using Hölder's inequality, we bound (2.14) as

$$(2.14) \lesssim (\|u_x\|_\infty^2 + \|u\|_\infty\|u_{xx}\|_\infty)\|\partial_x^{2k+1}u\|_2^2. \quad (2.16)$$

To estimate the summand in (2.14), we discuss several cases.

Case 1. $n = 2k + 1$. In view of the constraint, we have $(l, m) = (0, 2)$ or $(1, 1)$. Hölder's inequality then gives that

$$\left| \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1}u dx \right| \lesssim (\|u_x\|_\infty^2 + \|u\|_\infty\|u_{xx}\|_\infty)\|\partial_x^{2k+1}u\|_2^2.$$

Case 2. $n \leq 2k$, $l = 0$. Since $l + m + n = 2k + 3$ we must have $m \geq 3$. We choose $2 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and use Hölder to get

$$\left| \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1}u dx \right| \lesssim \|u\|_\infty \|\partial_x^m u\|_p \|\partial_x^n u\|_q \|\partial_x^{2k+1}u\|_2.$$

Using the interpolation inequality

$$\|\partial_x^m u\|_p \leq \|\partial_x^{2k+1}u\|_2^\theta \|u\|_{\dot{H}^{\frac{5}{2}}}^{1-\theta}, \quad \theta = \frac{m - 2 - \frac{1}{p}}{2k - \frac{3}{2}},$$

we have

$$\begin{aligned} \left| \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1}u dx \right| &\leq \|u\|_\infty \|\partial_x^{2k+1}u\|_2^2 \|u\|_{\dot{H}^{\frac{5}{2}}} \\ &\leq \|u\|_\infty \|u\|_{H^3} \|\partial_x^{2k+1}u\|_2^2. \end{aligned}$$

Case 3. $n \leq 2k$, $l = 1$. In this case $m, n \geq 2$, we can choose $2 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and use the interpolation inequality

$$\|\partial_x^m u\|_p \leq \|\partial_x^{2k+1} u\|_2^{\frac{m-1-\frac{1}{p}}{2k-\frac{1}{2}}} \|u\|_{\dot{H}^{\frac{3}{2}}}^{1-\frac{m-1-\frac{1}{p}}{2k-\frac{1}{2}}} \quad (2.17)$$

to get

$$\begin{aligned} \left| \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1} u dx \right| &\leq \|u_x\|_\infty \|\partial_x^m u\|_p \|\partial_x^n u\|_q \|\partial_x^{2k+1} u\|_2 \\ &\leq \|u_x\|_\infty \|\partial_x^{2k+1} u\|_2^2 \|u\|_{\dot{H}^{\frac{3}{2}}} \\ &\leq \|u_x\|_\infty \|u\|_{H^2} \|\partial_x^{2k+1} u\|_2^2. \end{aligned}$$

Case 4. $n \leq 2k$, $l, m, n \geq 2$. Choosing $2 < p, q, r < \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and using the interpolation inequality (2.17), we get

$$\begin{aligned} \left| \int \partial_x^l u \partial_x^m u \partial_x^n u \partial_x^{2k+1} u dx \right| &\leq \|\partial_x^l u\|_p \|\partial_x^m u\|_q \|\partial_x^n u\|_r \|\partial_x^{2k+1} u\|_2 \\ &\leq \|u\|_{\dot{H}^{\frac{3}{2}}}^2 \|\partial_x^{2k+1} u\|_2^2 \\ &\leq \|u\|_{H^2}^2 \|\partial_x^{2k+1} u\|_2^2. \end{aligned}$$

Collecting all the estimates together, we conclude

$$\begin{aligned} I &\leq (\|\partial_x u\|_\infty^2 + \|u\|_\infty \|u_{xx}\|_\infty + \|u\|_\infty \|u\|_{H^3} \\ &\quad + \|u_x\|_\infty \|u\|_{H^2} + \|u\|_{H^2}^2) \|\partial_x^{2k+1} u\|_2^2. \end{aligned}$$

For term II , we simply have

$$II = -\epsilon \int (\partial_x^{2k+2} u)^2 dx \leq 0.$$

For III, we compute:

$$\begin{aligned}
III &= - \int \partial_x^{2k+2} (K * \partial_x u) u \partial_x^{2k+1} u dx - \int K * \partial_x u \partial_x^{2k+2} u \partial_x^{2k+1} u dx \\
&\quad + \sum_{\substack{l+m=2k+2 \\ l,m \leq 2k+1}} C_{l,m} \int \partial_x^l (K * \partial_x u) \partial_x^m u \partial_x^{2k+1} u dx \\
&= - \int \partial_{xx} K * \partial_x^{2k+1} u u \partial_x^{2k+1} u dx + \frac{1}{2} \int \partial_x K * \partial_x u (\partial_x^{2k+1} u)^2 dx \\
&\quad + \sum_{\substack{l+m=2k+2 \\ l,m \leq 2k+1}} C_{l,m} \int \partial_x K * \partial_x^l u \partial_x^m u \partial_x^{2k+1} u dx \\
&= 2 \int u (\partial_x^{2k+1} u)^2 dx - \int (e^{-|x|} * \partial_x^{2k+1} u) \partial_x^{2k+1} u u dx + \frac{1}{2} \int (\partial_x K * \partial_x u) (\partial_x^{2k+1} u)^2 dx \\
&\quad + \sum_{\substack{l+m \leq 2k+2 \\ l,m \leq 2k+1}} C_{l,m} \int \partial_x K * \partial_x^l u \partial_x^m u \partial_x^{2k+1} u dx \\
&\lesssim (\|u\|_\infty + \|\partial_x u\|_2) \|\partial_x^{2k+1} u\|_2^2 \\
&\quad + \sum_{\substack{l+m=2k+2 \\ l,m \leq 2k+1}} C_{l,m} \|\partial_x^{2k+1} u\|_2 \|\partial_x^l u\|_2 \|\partial_x^m u\|_2.
\end{aligned}$$

Using the interpolation inequality

$$\|\partial_x^l u\|_2 \leq \|\partial_x^{2k+1} u\|_2^{\frac{l-1}{2k}} \|u_x\|_2^{1-\frac{l-1}{2k}},$$

we finally get

$$III \lesssim (\|u\|_\infty + \|u_x\|_2) \|\partial_x^{2k+1} u\|_2^2.$$

Summarizing the estimates of I, II, III we obtain

$$\begin{aligned}
\frac{d}{dt} \|\partial_x^{2k+1} u(t)\|_2^2 &\lesssim \|\partial_x^{2k+1} u\|_2^2 (\|u_x\|_\infty^2 + \|u\|_\infty \|u_{xx}\|_\infty \\
&\quad + \|u\|_\infty \|u\|_{H^3} + \|u_x\|_\infty \|u\|_{H^2} + \|u\|_\infty + \|u_x\|_2) \\
&\lesssim \|\partial_x^{2k+1} u\|_2^2 (\|u\|_{H^3} + \|u\|_{H^3}^2).
\end{aligned}$$

Using Gronwall and (2.7), (2.12), (2.13) we get

$$\sup_{0 \leq t \leq T_0} \|\partial_x^{2k+1} u(t)\|_2 \lesssim e^{CT_0} \|\partial_x^{2k+1} u_0\|_2,$$

for sufficiently small $T_0 = T_0(\|u_0\|_2, \|\partial_x u_0\|_2)$.

This concludes the estimate of the H_x^{2k+1} -norm of u^ϵ . Finally if $u_0 \geq 0$, then by the weak maximum principle we have $u^\epsilon(t) \geq 0$ for any $0 \leq t \leq T_0$. The lemma is proved. \square

We are now ready to complete the

Proof of Proposition 2.1. This follows directly from Lemma 2.5 and 2.6. In particular note that in Lemma 2.6 the time of existence of the local solution does not depend on ϵ . \square

Proof of Corollary 2.2. Assume $u_0 \in \bigcap_{m=0}^{\infty} H_x^m(\mathbb{R}) \cap L_x^p(\mathbb{R})$ for some $1 \leq p < 2$. Then using (2.1) and Duhamel's formula, we can write $u^\epsilon(t)$ as

$$\begin{aligned} u^\epsilon(t) &= e^{\epsilon t \partial_{xx}} u_0 + \int_0^t e^{\epsilon(t-s) \partial_{xx}} \left(\frac{1}{3} \partial_{xx}((u^\epsilon)^3) \right) (s) \\ &\quad - \int_0^t e^{\epsilon(t-s) \partial_{xx}} \partial_x(u^\epsilon \partial_x K * u^\epsilon)(s) ds. \end{aligned}$$

Since $\|e^{\epsilon t \partial_{xx}} f\|_p \lesssim \|f\|_p$ for any $\epsilon > 0$, we can estimate the L_x^p norm of u^ϵ as

$$\begin{aligned} \|u^\epsilon(t)\|_p &\lesssim \|u_0\|_p + \int_0^t \left\| \frac{1}{3} \partial_{xx}((u^\epsilon)^3) - \partial_x((\partial_x K * u^\epsilon) u^\epsilon) \right\|_p ds \\ &\lesssim \|u_0\|_p + \int_0^t \|u^\epsilon(s)\|_{W^{2,3p}}^3 + \|u^\epsilon(s)\|_{W^{1,2p}}^2 ds \\ &\lesssim \|u_0\|_p + \int_0^t \|u^\epsilon(s)\|_{H^3}^3 + \|u^\epsilon(s)\|_{H^2}^2 ds, \end{aligned}$$

where the last inequality follows from the Sobolev embedding. This estimate shows that $u^\epsilon(t) \in L_x^p$ for any t . The continuity (including right continuity at $t = 0$) follows from similar estimates. We omit the details. Finally since for $1 \leq p < 2$, $\|u_0\|_2 + \|\partial_x u_0\|_\infty \lesssim \|u_0\|_p + \|\partial_x u_0\|_\infty$, one can choose the time interval sufficiently small depending only on $\|u_0\|_p + \|\partial_x u_0\|_\infty$. \square

2.2. Proof of Proposition 2.3. In the case $u_0 \geq 0$, we shall show the regularized equation has a global solution. The key point here is that by using the positivity, we can obtain the apriori boundedness of the L_x^p -norm ($2 \leq p \leq \infty$) of the solution on any finite time interval. To control the L_x^∞ -norm of the gradient, we need the following lemma from [39].

Lemma 2.7. *Let $\Omega = (0, T) \times \mathbb{R}$ and $u \in C^{2,1}(\Omega) \cap C(\bar{\Omega})$ be a solution to the parabolic equation:*

$$\begin{cases} \partial_t u = b(t, x, u, u_x) u_{xx} + a(t, x, u, u_x), & (t, x) \in (0, T) \times \mathbb{R} \\ u|_{t=0} = u_0 \end{cases}$$

with the following properties:

- (1) *Uniform parabolicity: there exists constants $\beta_0, \beta_1 > 0$ such that*

$$\beta_0 b(t, x, z, p) p^2 \geq |a(t, x, z, p)|, \quad \forall |p| \geq \beta_1.$$

(2) *There is a constant $M > 0$ such that*

$$|u(t, x) - u(t, y)| \leq M, \quad \forall (t, x), (t, y) \in \Omega.$$

(3) *There exists $L_0 > 0$ such that*

$$|u_0(x) - u_0(y)| \leq L_0|x - y|, \quad \forall x, y \in \mathbb{R}.$$

Under all the above assumptions, we have the following bound:

$$\sup_{0 \leq t \leq T} \|u_x(t)\|_\infty \leq 2(L_0 + \beta_1)e^{\beta_0 M}.$$

Proof. See Lemma 11.16 of [39]. □

Incorporating Lemma 2.7 with the idea of the proof of Lemma 2.6, we get the following global analogue of Lemma 2.6.

Lemma 2.8. *Let $u_0 \in H_x^{2k+1}(\mathbb{R})$, $k \geq 2$ and $u_0 \geq 0$. Then for any $T > 0$ and for any $\epsilon > 0$, (2.1) has a unique solution $u^\epsilon \in B_T^k$. Also $u^\epsilon(t) \geq 0$ for any $t \geq 0$.*

Proof. The positivity of u^ϵ follows easily from the weak maximum principle. By Lemma 2.5 we only need to get an apriori control of the H_x^{2k+1} -norm of u on an arbitrary time interval $[0, T]$.

Step 1: L_x^p -norm control for $2 \leq p \leq +\infty$. By the same estimates as in Step 1 of the proof of Lemma 2.6 we have the following a priori L_x^2 -norm estimate:

$$\|u(t)\|_2 \leq \|u_0\|_2 \exp\left(\frac{1}{4}\|\partial_x K\|_1^2 t\right).$$

To obtain the L_x^∞ -norm estimate, we take $2 < p < +\infty$ and compute

$$\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p \leq -\frac{2(p-1)}{p} \int u|u|^p dx + \frac{p-1}{p} \int (e^{-|x|} * u)|u|^p dx$$

Since $u \geq 0$ we can drop the first term and continue to estimate

$$\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p \leq C\|u_0\|_2 e^{Ct} \|u(t)\|_p^p,$$

where C is a generic constant independent of p . Gronwall then implies that

$$\|u(t)\|_p \leq \|u_0\|_p \exp(Ct\|u_0\|_2 e^{Ct}). \quad (2.18)$$

Letting $p \rightarrow \infty$, we obtain

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \exp(Ct\|u_0\|_2 e^{Ct}). \quad (2.19)$$

This is the L_x^∞ -norm estimate we needed.

Step 2: Control of $\|\partial_x u(t)\|_\infty$. We shall apply Lemma 2.7 with

$$b(t, x, z, p) = z^2 + \epsilon,$$

$$a(t, x, z, p) = 2zp^2 + 2z^2 - c(t, x)z - d(t, x)p,$$

where

$$\begin{aligned} c(t, x) &= (\partial_x K * u_x)(t, x), \\ d(t, x) &= (\partial_x K * u)(t, x). \end{aligned}$$

By step 1 we have

$$\begin{aligned} |c(t, x)| + |d(t, x)| &\leq C\|u(t)\|_\infty \\ &\leq C\|u_0\|_\infty \exp(CT\|u_0\|_2 e^{CT}). \end{aligned}$$

Also

$$\begin{aligned} |u(t, x) - u(t, y)| &\leq 2\|u(t)\|_\infty \leq 2\|u_0\|_\infty \exp(CT\|u_0\|_2 e^{CT}) \\ &=: M \end{aligned}$$

and

$$\begin{aligned} |u_0(x) - u_0(y)| &\leq \|\partial_x u_0\|_\infty |x - y| \\ &=: L_0 |x - y|. \end{aligned}$$

Collecting all these estimates, we see that

$$\sup_{0 \leq t \leq T} \|\partial_x u(t)\|_\infty \leq C(\|u_0\|_2, \|\partial_x u_0\|_\infty, \epsilon, T)$$

This concludes the gradient estimate.

Step 3: Control of the higher derivatives. This part of the estimates is exactly the same as the corresponding estimates in the proof of Lemma 2.6. Note in particular that we do not need T to be small once we obtain the a priori control of $\|u\|_p$ and $\|\partial_x u\|_\infty$ norms. \square

Now we are ready to complete

Proof of Proposition 2.3. This follows directly from the local existence Lemma 2.5 and the a priori estimate Lemma 2.8. If $u_0 \in L_x^p(\mathbb{R})$ for some $1 \leq p < 2$, then by repeating the proof of Corollary 2.2 we obtain $u \in C([0, T], L_x^p)$ for any $T > 0$. The proof is finished. \square

3. PROOF OF THEOREM 1.1, 1.3 AND 1.4

In this section we prove our main theorems. Our solution is going to be the limit of the sequence of regularized solutions u^ϵ which we constructed in the previous section. To obtain uniform control of the Sobolev norms of the regularized solutions, we have the following

Lemma 3.1 (A priori estimate). *Assume $u_0 \in H_x^{2k+1}(\mathbb{R})$, $k \geq 2$. Let $T_0 = T_0(\|u_0\|_2 + \|\partial_x u_0\|_\infty)$ and $u^\epsilon \in B_{T_0}^k$ be the corresponding unique solution to (2.1) (see Lemma 2.6). Then the set of functions $(u^\epsilon)_{0 < \epsilon < 1}$*

satisfies the following uniform estimates:

$$\begin{aligned} \sup_{0 \leq t \leq T_0} \|u^\epsilon\|_{H^{2k+1}} + \sqrt{\epsilon} \|u^\epsilon\|_{L_t^2 H^{2k+2}([0, T_0] \times \mathbb{R})} &\leq C(\|u_0\|_{H^{2k+1}}, k), \\ \|u^\epsilon \partial_x^{2k+2} u^\epsilon\|_{L_t^2 L_x^2([0, T_0] \times \mathbb{R})} + \|\partial_t u^\epsilon\|_{L_t^2 H^{2k}([0, T_0] \times \mathbb{R})} &\leq C(\|u_0\|_{H^{2k+1}}, k), \\ \|\partial_x(\partial_x K * u^\epsilon u^\epsilon)\|_{L_t^2 H^{2k}([0, T_0] \times \mathbb{R})} &\leq C(\|u_0\|_{H^{2k+1}}, k), \end{aligned}$$

where $C = C(\|u_0\|_{H^{2k+1}}, k)$ is a positive constant.

Proof. The estimates of $\|u^\epsilon\|_{L_t^\infty H^{2k+1}}$, $\sqrt{\epsilon} \|u^\epsilon\|_{L_t^2 H^{2k+2}}$ and $\|u^\epsilon \partial_x^{2k+2} u^\epsilon\|_{L_t^2 L_x^2}$ can be recovered from step 4 of the proof of Lemma 2.6. To bound $\|\partial_t u^\epsilon\|_{L_t^2 H^{2k}}$, we use (2.1) to obtain (here we again drop the superscript ϵ for simplicity):

$$\begin{aligned} \|\partial_t u\|_{H^{2k}} &\leq \|\partial_t u\|_2 + \|\partial_x^{2k} \partial_t u\|_2 \\ &\leq \|(u^2 + \epsilon)u_{xx} + 2uu_x^2 - \partial_x(K * \partial_x uu)\|_2 + \|\partial_x^{2k}(u^2 u_{xx})\|_2 \\ &\quad + \epsilon \|\partial_x^{2k+2} u\|_2 + 2\|\partial_x^{2k}(uu_x^2)\|_2 + \|\partial_x^{2k+1}(K * \partial_x uu)\|_2 \\ &\leq \|u\|_{H^2}^3 + \|u\|_{H^2}^2 + \|u^2 \partial_x^{2k+2} u\|_2 + \|u\|_{H^{2k+1}}^3 \\ &\quad + \epsilon \|\partial_x^{2k+2} u\|_2 + \|u\|_{H^{2k+1}}^3 + \|u\|_{H^{2k+1}}^2. \end{aligned}$$

Integrating over $[0, T_0]$, we have

$$\begin{aligned} \int_0^{T_0} \|\partial_t u(t)\|_{H^{2k}}^2 dt &\lesssim T_0 \sup_{0 \leq t \leq T_0} \|u\|_{H^{2k+1}}^2 (1 + \|u\|_{H^{2k+1}}) \\ &\quad + \int_0^{T_0} \|u^2 \partial_x^{2k+2} u(t)\|_2^2 dt + \epsilon \int_0^T \|\partial_x^{2k+2} u\|_2^2 dt \\ &\leq C(T_0, k, \|u_0\|_{H^{2k+1}}). \end{aligned}$$

Since $T_0 = T_0(\|u_0\|_2 + \|\partial_x u_0\|_\infty)$, we obtain the desired bounds. The last estimate is a simple application of Hölder and Young's inequality using the given estimates. \square

Now we define the set

$$\begin{aligned} D_{T_0}^k := &\left\{ v \in C([0, T_0]; H_x^{2k}) \cap L^\infty(0, T_0; H_x^{2k+1}) : \right. \\ &\left. \partial_t v \in L^2(0, T_0; H^{2k}), v^2 \partial_{xx} v \in L^2(0, T_0; H^{2k}) \right\}. \end{aligned}$$

We shall prove the existence and the uniqueness of the solution of (1.1) in this set. This is

Theorem 3.2. *Assume $u_0 \in H^{2k+1}(\mathbb{R})$ with $k \geq 2$. Then there exists $T_0 = T_0(\|u_0\|_2 + \|\partial_x u_0\|_\infty)$ such that (1.1) has a unique solution $u \in D_{T_0}^k$.*

Proof. Step 1. Contraction in $C([0, T], L_x^2)$. For any two $\epsilon_1, \epsilon_2 > 0$, let $u = u^{\epsilon_1}$, $v = u^{\epsilon_2}$ solve (2.1) with the same initial data u_0 . Denote

$w = u - v$, then for w we have the equation

$$\begin{aligned}\partial_t w &= \frac{1}{3}(u^3 - v^3)_{xx} + \epsilon_1 u_{xx} - \epsilon_2 v_{xx} - \partial_x(K * \partial_x w u) \\ &\quad - \partial_x(K * \partial_x v w).\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 &= \frac{1}{3} \int (u^3 - v^3)_{xx} w dx + \int (\epsilon_1 u_{xx} - \epsilon_2 v_{xx}) w dx \\ &\quad - \int \partial_x(K * \partial_x w u) w dx - \int \partial_x(K * \partial_x v w) w dx \\ &=: I + II + III + IV.\end{aligned}$$

Denote $F(u, v) = u^2 + v^2 + uv$. Clearly we always have $F(u, v) \geq 0$. We compute

$$\begin{aligned}I &= \frac{1}{3} \int \partial_{xx}(wF(u, v)) w dx \\ &= \frac{1}{3} \int \partial_{xx} w w F(u, v) dx \\ &= -\frac{1}{3} \int (\partial_x w)^2 F(u, v) dx + \frac{1}{6} \int w^2 \partial_{xx} F(u, v) dx \\ &\leq \frac{1}{6} \int w^2 \partial_{xx} F(u, v) dx \\ &\lesssim \|w\|_2^2 (\|u\|_{H^3}^2 + \|v\|_{H^3}^2).\end{aligned}$$

For II we have the estimate

$$II \leq (\epsilon_1 \|u\|_{H^2} + \epsilon_2 \|v\|_{H^2}) \|w\|_2.$$

The term III can be bounded as:

$$\begin{aligned}III &= \int K * \partial_x w u w_x dx \\ &= - \int \partial_{xx} K * w u w dx - \int K * \partial_x w u_x w dx \\ &\leq (\|u\|_\infty + \|\partial_x u\|_\infty) \|w\|_2^2.\end{aligned}$$

Similarly for IV we get

$$\begin{aligned}IV &\lesssim \|K * \partial_{xx} v\|_\infty \|w\|_2^2 \\ &\lesssim \|v\|_{H^1} \|w\|_2^2.\end{aligned}$$

Finally we have by Lemma 3.1

$$\begin{aligned}\frac{d}{dt} \|w(t)\|_2^2 &\lesssim \|w(t)\|_2^2 (\|u\|_{H^3}^2 + \|v\|_{H^3}^2 + \|u\|_{H^3} + \|v\|_{H^3}) \\ &\quad + (\epsilon_1 \|u\|_{H^2} + \epsilon_2 \|v\|_{H^2}) \|w(t)\|_2 \\ &\lesssim C(\|u_0\|_{H^{2k+1}}, k) \|w(t)\|_2^2 + C(\|u_0\|_{H^{2k+1}}, k) (\epsilon_1 + \epsilon_2) \|w(t)\|_2\end{aligned}$$

Gronwall then gives

$$\begin{aligned} \|u^{\epsilon_1}(t) - u^{\epsilon_2}(t)\|_2 &= \|w(t)\|_2 \\ &\leq C(\|u_0\|_{H^{2k+1}}, k)(\epsilon_1 + \epsilon_2) \exp(C(\|u_0\|_{H^{2k+1}}, k)t). \end{aligned}$$

This implies that $(u^\epsilon)_{0 < \epsilon < 1}$ has a limit as $\epsilon \rightarrow 0$ in $C([0, T_0]; L_x^2)$. Denote this limit as $u \in C([0, T_0]; L_x^2)$.

Step 2. We show that u is our desired solution in $D_{T_0}^k$. By Lemma 3.1 and the interpolation inequality

$$\|f\|_{H^{2k}} \lesssim \|f\|_2^{\frac{1}{2k+1}} \|f\|_{H^{2k+1}}^{\frac{2k}{2k+1}},$$

we obtain

$$u^\epsilon \rightarrow u \quad \text{in } C([0, T_0]; H_x^{2k}), \quad \text{as } \epsilon \rightarrow 0.$$

It then follows easily that

$$\begin{aligned} (u^\epsilon)^2 \partial_{xx} u^\epsilon &\rightarrow u^2 \partial_{xx} u, \quad \text{in } C(0, T_0; L_x^2), \\ u^\epsilon (\partial_x u^\epsilon)^2 &\rightarrow u (\partial_x u)^2, \quad \text{in } C(0, T_0; L_x^2), \\ \partial_x (K * \partial_x u^\epsilon u^\epsilon) &\rightarrow \partial_x (K * \partial_x u u), \quad \text{in } C(0, T_0; L_x^2). \end{aligned}$$

By Lemma 3.1, the functions $(u^\epsilon)^2 \partial_{xx} u^\epsilon$, $u^\epsilon (\partial_x u^\epsilon)^2$, $\partial_x (K * \partial_x u^\epsilon u^\epsilon)$ are uniformly bounded in $L^2([0, T_0], H_x^{2k})$. We conclude that $u^2 \partial_{xx} u$, $u (\partial_x u)^2$, $\partial_x (K * \partial_x u u) \in L^2([0, T_0], H_x^{2k})$. Now since u^ϵ is also uniformly bounded in $L^\infty([0, T_0], H_x^{2k+1})$, we obtain $u \in L^\infty([0, T_0], H_x^{2k+1})$. Similarly by Lemma 3.1, the set of functions $(\partial_t u^\epsilon)_{0 < \epsilon < 1}$ is uniformly bounded in $L^2([0, T_0], H_x^{2k})$. By extracting a subsequence if necessary, we obtain for some $\epsilon_n \rightarrow 0$,

$$\partial_t u^{\epsilon_n} \rightharpoonup \partial_t u, \quad \text{in } L^2([0, T_0], H_x^{2k}).$$

It follows that $\partial_t u \in L^2([0, T_0], H_x^{2k})$. Finally by using Lemma 3.1 again we have

$$\epsilon_n \partial_{xx} u^{\epsilon_n} \rightarrow 0, \quad \text{in } L^2(0, T_0; H_x^{2k}).$$

This shows that u is our desired solution in $D_{T_0}^k$. The theorem is proved. \square

Corollary 3.3 (Continuation of solutions). *Let $u_0 \in H_x^{2k+1}(\mathbb{R})$ with $k \geq 2$. Then there exists a unique solution u of (1.1) with maximal-lifespan T^* such that only one of the following possibilities occur:*

- (1) $T^* = \infty$ and $u \in D_T^k$ for any $T > 0$.
- (2) $T^* < \infty$, $u \in D_T^k$ for any $T < T^*$, and

$$\lim_{t \rightarrow T^*} \|\partial_x u(t)\|_\infty = \infty.$$

For any $2 \leq p < \infty$, there exists a generic constant C such that

$$\|u(t)\|_p \leq \|u_0\|_p e^{Cpt}, \quad \forall t \geq 0.$$

If in addition $u_0 \geq 0$, then $u(t) \geq 0$ and we also have the p -independent estimate for all $2 \leq p \leq +\infty$:

$$\|u(t)\|_p \leq \|u_0\|_p \exp(Ct\|u_0\|_2 e^{Ct}), \quad \forall t \geq 0. \quad (3.1)$$

In particular if $p = +\infty$, then

$$\|u(t)\|_\infty \leq \|u_0\|_\infty \exp(Ct\|u_0\|_2 e^{Ct}), \quad \forall t \geq 0.$$

Proof. This follows directly from Theorem 3.2 and Lemma 2.6 (see (2.5) for the growth estimate of L_x^p -norm for any $T > 0$). Since $u^\epsilon \rightarrow u$ uniformly in $C([0, T]; H_x^{2k})$, $u(t)$ satisfies the same estimate (2.5). Similarly if $u_0 \geq 0$, then the estimate (3.1) follows from the corresponding estimate for u^ϵ (see (2.18) and (2.19) in the proof of Lemma 2.8). Now that the L^2 norm of the constructed local solution remains finite for any $T > 0$, it can be continued as long as $\|\partial_x u(t)\|_\infty$ does not blow up. \square

We are now ready to complete the

Proof of Theorem 1.1. By Theorem 3.2, $u_0 \in \bigcap_{m=0}^\infty H_x^m(\mathbb{R})$ implies that $u \in \bigcap_{k=2}^\infty D_{T_0}^k$. Therefore $u \in C([0, T_0], H_x^m)$ for any $m \geq 0$. It also follows that $\partial_t u \in \bigcap_{m=1}^\infty L^2(0, T_0; H_x^m)$. By using (1.1), a standard bootstrap argument implies that $u \in C^\infty([0, T_0] \times \mathbb{R})$. If $u_0 \geq 0$, then the positivity of u follows from Lemma 2.8 and the uniform convergence of u^ϵ to u . The theorem is proved. \square

Proof of Theorem 1.3. This follows immediately from Theorem 1.1 and Corollary 3.3. \square

Proof of Theorem 1.4. This is only slightly different from the proof of Corollary 2.2. Assume $u_0 \in \bigcap_{m=0}^\infty H_x^m(\mathbb{R}) \cap L_x^p(\mathbb{R})$ for some $1 \leq p < 2$. Then using (1.1), we can bound

$$\begin{aligned} \|u(t)\|_p &\lesssim \|u_0\|_p + \int_0^t \left\| \frac{1}{3} \partial_{xx}(u^3) - \partial_x((\partial_x K * u)u) \right\|_p ds \\ &\lesssim \|u_0\|_p + \int_0^t \|u(s)\|_{W^{2,3p}}^3 + \|u(s)\|_{W^{1,2p}}^2 ds \\ &\lesssim \|u_0\|_p + \int_0^t \|u(s)\|_{H^3}^3 + \|u(s)\|_{H^2}^2 ds. \end{aligned}$$

This shows that $u(t) \in L_x^p$ for any t . The continuity (including right continuity at $t = 0$) follows from similar estimates. We omit the standard details. Finally in the case $u_0 \geq 0$, $u_0 \in L_x^1(\mathbb{R})$, the L_x^1 preservation follows from direct integration. \square

4. PROOF OF THEOREM 1.5

We argue by contradiction. Let $u_0 \in C_c^\infty(\mathbb{R})$, $u_0 \geq 0$ and assume that the corresponding solution u is global. By Theorem 1.1, 1.3, 1.4, $u \in C([0, T], H_x^m)$ for any $m \geq 0$ and the L_x^1 -norm of u is preserved. Our intuition of proving the blowup is based on the observation that as time goes on, the boundary of the solution (which in 1D case consists of two points) will move face to face at a certain speed which has a lower bound independent of time. This clearly will lead to the collapse of the solution. To realize this intuition, we will carry out a detailed analysis on the characteristic lines of the solution which satisfy

$$\begin{cases} \frac{d}{dt}X(t, \alpha) = (K * \partial_x u)(X(t, \alpha), t), \\ X(0, \alpha) = \alpha \in \mathbb{R}. \end{cases} \quad (4.1)$$

By standard ODE theory and our assumption that u is a smooth global solution, $X(t)$ is well defined and smooth for all time. Moreover, we have the following lemma.

Lemma 4.1 (Properties of $X(t, \alpha)$). *The characteristic lines $X(t, \alpha)$ and the solution $u(t, x)$ satisfy the following properties:*

- (1) *For any $t \geq 0$, $X(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a C^∞ diffeomorphism.*
- (2) *$X(t, \cdot)$ is an order-preserving map in the sense that if $\alpha_1 < \alpha_2$, then $X(t, \alpha_1) < X(t, \alpha_2)$ for any $t \geq 0$.*
- (3) *$X(t, \cdot)$ maps intervals to intervals. More precisely, for any $\alpha_1 < \alpha_2$, denote*

$$X(t, \cdot)([\alpha_1, \alpha_2]) = \{X(t, \alpha) : \alpha_1 \leq \alpha \leq \alpha_2\}.$$

Then $X(t, \cdot)([\alpha_1, \alpha_2]) = [X(t, \alpha_1), X(t, \alpha_2)]$.

- (4) *If $u_0(\alpha) = 0$, then $u(X(t, \alpha), t) = 0$ for any $t \geq 0$.*
- (5) *If $\text{supp}(u_0(\cdot)) \subset [-L, L]$ for some $L > 0$, then*

$$\text{supp}(u(t, \cdot)) \subset [X(t, -L), X(t, L)] \subset [-L, L]$$

for all $t \geq 0$.

Proof. Property (1) is rather standard. Note in particular that

$$\frac{\partial X(t, \alpha)}{\partial \alpha} = \exp \left(\int_0^t (\partial_x K * \partial_x u)(X(s, \alpha), s) ds \right)$$

is smooth and globally invertible due to the fact that u is smooth. Property (2) is also trivial. Assume not true, then by the intermediate value theorem, for some $\tau > 0$ we must have $X(\tau, \alpha_1) - X(\tau, \alpha_2) = 0$ which is a contradiction to property (1). Property (3) follows immediately from property (1) and (2). For property (4), one simply uses (1.1) and (4.1). Writing $u(t, \alpha) = u(X(t, \alpha), t)$, we have

$$\begin{cases} \frac{d}{dt}u(t, \alpha) = (-\partial_x K * \partial_x u + u \partial_{xx} u + 2(\partial_x u)^2) u(t, \alpha), \\ u(0, \alpha) = u_0(\alpha). \end{cases}$$

On the RHS of the above equation, one can regard the coefficient of $u(t, \alpha)$ as given functions of (t, α) . By direct integration, property (4) now follows immediately. For property (5) we take $y < x(t, -L)$, then by property (1) and (2) we have $y = X(t, \alpha)$ for some $\alpha < -L$. By property (4) since $u_0(\alpha) = 0$, we must have $u(t, y) = u(t, X(t, \alpha)) = 0$. Similarly one can show that if $y > X(t, L)$ then $u(t, y) = 0$. This shows that $\text{supp}(u(t, \cdot)) \subset [X(t, -L), X(t, L)]$. Now using this fact and (4.1), we compute

$$\begin{aligned}
\frac{d}{dt}X(t, -L) &= (\partial_x K * u)(t, X(t, -L)) \\
&= - \int_{\mathbb{R}} \text{sign}(X(t, -L) - y) e^{-|X(t, -L) - y|} u(t, y) dy \\
&= - \int_{y \geq X(t, -L)} \text{sign}(X(t, -L) - y) e^{-|X(t, -L) - y|} u(t, y) dy \\
&\geq 0.
\end{aligned} \tag{4.2}$$

By a similar computation one can show that $\frac{d}{dt}X(t, L) \leq 0$. These facts together with property 2 and 3 above easily yield that $[X(t, -L), X(t, L)] \subset [-L, L]$. \square

Using lemma 4.1, we are able to finish the

Proof of Theorem 1.5. We argue by contradiction. Let $u_0 \in C_c^\infty(\mathbb{R})$, $u_0 \geq 0$ with $\text{supp}(u_0) \subset [-L, L]$ for some $L > 0$. Assume that the corresponding solution u is global. Define the characteristic $X(t, \alpha)$ according to (4.1). By Lemma 4.1 we have

$$\text{supp}(u(t, \cdot)) \subset [X(t, -L), X(t, L)] \subset [-L, L], \quad \forall t \geq 0. \tag{4.3}$$

By this fact, we then compute using (4.2),

$$\begin{aligned}
\frac{d}{dt}X(t, -L) &= - \int_{y \geq X(t, -L)} \text{sign}(X(t, -L) - y) e^{-|X(t, -L) - y|} u(t, y) dy \\
&= - \int_{X(t, -L) \leq y \leq X(t, L)} \text{sign}(X(t, -L) - y) e^{-|X(t, -L) - y|} u(t, y) dy \\
&= \int_{X(t, -L) \leq y \leq X(t, L)} e^{-|X(t, -L) - y|} u(t, y) dy \\
&\geq e^{-2L} \|u_0\|_{L^1},
\end{aligned}$$

where in the last step we used the fact that the L^1 norm of u is preserved. This shows that $X(t, -L)$ grows linearly with time which is contradiction to (4.3). Thus we have shown that the solution u with u_0 as initial data cannot exist globally. Finally (1.4) and (1.5) are easy consequences of Theorem 1.1, 1.3 and 1.4. The theorem is proved. \square

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